AN ALGORITHM FOR FINDING THE VEECH GROUP OF AN ORIGAMI

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Abstract. We study the Veech group of an origami, i.e. of a translation surface, tessellated by parallelograms. We show that it is isomorphic to the image of a certain subgroup of $\mathrm{Aut}^+(F_2)$ in $\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Out}^+(F_2)$. Based on this we present an algorithm that determines the Veech group.

1. Origamis as Teichmüller curves

(Oriented) origamis (as defined in section 2.1) can be described as follows: Take finitely many copies of the unit square in $\mathbb{C}$ and glue them together such that left edges are glued with right edges and upper edges with lower ones (compare [Lochak 2003], [Möller 2003]). This defines a compact surface S. We restrict ourselves to the cases where S is connected.

Lifting the structure of $\mathbb{C}$ via the squares defines a translation structure on $S^\star := S - \{P_1, \ldots, P_n\}$, where $P_1, \ldots, P_n$ are finitely many points on S. One can vary the structure on $S^\star$ as follows: For each $\tau \in \mathbb{H}$ identify the squares on S with the parallelogram $P(\tau)$ in $\mathbb{C}$ defined by the vertices $0, 1, \tau, 1+\tau$. This defines an isometric embedding of the upper half plane $\mathbb{H}$ into the Teichmüller space $T_{g,n}$, where g is the genus of S. This embedding is described in detail in [Lochak 2003] and [McMullen 2003] in the more general context of Teichmüller curves. The image of $\mathbb{H}$ in $T_{g,n}$ under this embedding is a complex geodesic $\Delta \subset T_{g,n}$. The image C of Δ in the moduli space $M_{g,n}$ under the natural projection $T_{g,n} \to M_{g,n}$ is birational to the mirror image of $\mathbb{H}/\Gamma(O)$ ([Lochak 2003], [McMullen 2003]), where $\Gamma(O)$ is the Veech group of an origami O, defined as in section 2.1.

$\mathbb{H}/\Gamma(O)$ is an algebraic curve defined over $\bar{\mathbb{Q}}$ (see 3.4). One has even more: The embedded curve C in $M_{g,n}$ is an irreducible component of a Hurwitz space and thus also defined over $\overline{\mathbb{Q}}$ ([Möller 2003]). In [Lochak 2003], where origamis were originally introduced, Pierre Lochak suggests to study them in the context of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on combinatorial objects, in some sense as generalization of the study of dessins d'enfants. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of *origami curves* in $M_{g,n}$, and this action is faithful as shown in [Möller 2003].

Origami curves represent a special kind of (imprimitive) Teichmüller curves, described in [McMullen 2003], namely those that arise via a torus.

In this article we study the Veech group $\Gamma(O)$ of origamis O. We describe an algorithm that finds generators and coset representatives of $\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$ and calculates the genus and the number of points at infinity of $\mathbb{H}/\Gamma(O)$.

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2. Veech groups of origamis

The algorithm we want to present is based on the following Proposition 1. We denote by F_2 the free group in two generators and by $\mathrm{Aut}^+(F_2)$ the group of orientation preserving automorphisms of F_2. Furthermore, we use the fact that $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to $\mathrm{Out}^+(F_2)$, the group of outer orientation preserving automorphisms of F_2, and denote by $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{Out}^+(F_2) \cong \mathrm{SL}_2(\mathbb{Z})$ the canonical projection (see Lemma 8).
To an origami $O := (p : X \to E^\star)$ we will associate a subgroup $H \cong \mathrm{Gal}(\mathbb{H}/X)$ of F_2. $\Gamma(O)$ is the Veech group of O.

Proposition 1. *Let O be an origami. Let* $\mathrm{Aff}^+(H) := \{\gamma \in \mathrm{Aut}^+(F_2) | \gamma(H) = H\}$. *Then we have:*

$$\Gamma(O) = \hat{\beta}(\mathrm{Aff}^+(H)) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

The aim of Section 2 is to explain the notations and prove the statement of Proposition 1.

2.1. **Origamis, translation surfaces and the Veech Group.**

In the following let E be a fixed torus and $E^\star := E - \{\bar{P}\}$ (for some $\bar{P} \in E$) be a once punctured torus.

Definition 2. An (oriented) *origami* O (of genus $g \geq 1$) is a (topological) unramified covering $p : X \to E^\star$, where X is obtained by erasing finitely many points of a compact surface $\bar{X}$ of genus g.

Fix a (topological) unramified universal covering $u : \tilde{X} \to X$ of X. Then $v := p \circ u$ is a universal covering of $E^\star$.
Let $\mathrm{Gal}(\tilde{X}/E^\star)$ be the group of its deck transformations. It is naturally isomorphic to the fundamental group $\pi_1(E^\star, \bar{Q})$ of $E^\star$ with an arbitrary base point $\bar{Q} \in E^\star$. Furthermore, $\pi_1(E^\star, \bar{Q})$ is isomorphic to $F_2 := F_2(x, y)$, the free group in the two generators x and y. Fix this isomorphism $\alpha : F_2 \to \pi_1(E^\star, \bar{Q}) \overset{\text{can}}{\cong} \mathrm{Gal}(\tilde{X}/E^\star)$ such that $\alpha(x)$ and $\alpha(y)$ define a canonical marking on $E^\star$.
Then, $H := \mathrm{Gal}(\tilde{X}/X) \subseteq \mathrm{Gal}(\tilde{X}/E^\star)$ is considered (via $(\mathrm{can} \circ \alpha)^{-1}$) as subgroup of F_2.

Notation 3.

$$H := \mathrm{Gal}(\tilde{X}/X) \subseteq \mathrm{Gal}(\tilde{X}/E^\star) = F_2(x, y) =: F_2.$$

We will consider translation structures on X induced by translation structures on $E^\star$. Therefore we first want to recall some definitions and notations (see [Thurston 1997], [Gutkin/Judge 2000]).

An atlas on a surface X such that all transition maps are translations defines a *translation structure* μ on X. $X_\mu := (X, \mu)$ is called *translation surface*. We call

$$\mathrm{Aff}^+(X_\mu) := \{f : X_\mu \to X_\mu | f \text{ is an orientation preserving affine diffeomorphism}^1\}$$

the affine group of X_μ.
Let $u : \tilde{X} \to X$ be a (topological) universal covering of X. Then $\tilde{X}$ becomes a translation surface $\tilde{X}_\eta$ by lifting the structure μ on X via u to η on $\tilde{X}$. A fixed chart (U, η_U) of $\tilde{X}_\eta$ defines a holomorphic map $\mathbf{dev} : \tilde{X}_\eta \to \mathbb{C}$ (*developing map*) such that

$$\eta_U = dev|_U \quad \text{and} \quad \eta_{U'} = t \circ dev|_{U'} \text{ for a translation } t := t(U', \eta_{U'})$$

for any other chart $(U', \eta_{U'})$ of $\tilde{X}_\eta$.
For any affine diffeomorphism $\hat{f}$ of $\tilde{X}_\eta$ there is a unique affine diffeomorphism $\mathbf{aff}(\hat{f})$ of $\mathbb{C}$ such that $\mathbf{dev} \circ \hat{f} = \mathbf{aff}(\hat{f}) \circ \mathbf{dev}$. We call $\mathbf{aff}$ the group homomorphism

$$\mathbf{aff} : \mathrm{Aff}^+(\tilde{X}_\eta) \to \mathrm{Aff}^+(\mathbb{C}), \hat{f} \mapsto \mathbf{aff}(\hat{f}).$$

The *holonomy mapping* **hol** is the restriction of **aff** to the subgroup $H = \mathrm{Gal}(\tilde{X}/X)$ of $\mathrm{Aff}^+(\tilde{X})$. If proj is the natural projection $\mathrm{proj} : \mathrm{Aff}^+(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{R})$, then the group homomorphism

$$\mathbf{der} : \mathrm{Aff}^+(X_\mu) \to \mathrm{GL}_2(\mathbb{R}), f \mapsto \mathrm{proj}(\mathbf{aff}(\hat{f}))$$
$$\text{where } \hat{f} \text{ is some lift of } f \text{ to } \tilde{X}$$

is well defined and called *derived map* .

$\Gamma(X_\mu) := \mathbf{der}(\mathrm{Aff}^+(X_\mu)) \subseteq \mathrm{GL}_2(\mathbb{R})$ is called the *Veech group* of X_μ. It is independent of the choice of the chart (U, μ_U) which we used to define **dev**. If X is precompact, i.e. X is obtained by erasing finitely many points from a compact Riemann surface $\bar{X}$, then every $f \in \mathrm{Aff}^+(X_\mu)$ preserves the volume. Thus, $\Gamma(X_\mu)$ is in $\mathrm{SL}_2(\mathbb{R})$.

Now, given an origami $O = (p : X \to E^\star)$ as above, any matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

defines a translation structure on X as follows:
Take the lattice

$$\Lambda_B := < \vec{v}_1 := \begin{pmatrix} a \\ c \end{pmatrix}, \vec{v}_2 := \begin{pmatrix} b \\ d \end{pmatrix} > \text{ in } \mathbb{C}.$$

Let $E_B := \mathbb{C}/\Lambda_B$ be the elliptic curve defined by Λ_B and let $E_B^\star$ be the once punctured elliptic curve (obtained by erasing the image of 0 from E_B) with the induced translation structure. Fix some point Q_B in $\mathbb{C} - \Lambda_B$. Let $\bar{Q}_B$ be its image on $E_B^\star$. Fix furthermore as canonical marking the images of the segments from Q_B to $Q_B + \vec{v}_1$ and from Q_B to $Q_B + \vec{v}_2$ on $E_B^\star$. Identify $E_B^\star$ with $E^\star$ via a diffeomorphism respecting the canonical markings. This way p defines an unramified covering of $E_B^\star$. Let μ_B be the translation structure on X defined by lifting the translation structure on $E_B^\star$ to X via p (μ_B depends also on p !). Similarly let η_B be the translation structure on the fixed universal covering $\tilde{X}$ defined via u.

[1] In the following all diffeomorphisms are orientation preserving

Notation 4. Denote by $X_B := X_B(O) := (X, \mu_B)$ the surface X with translation structure μ_B. Furthermore, denote by $\tilde{X}_B$ the translation surface $(\tilde{X}, \eta_B)$.

Then the maps $p_B : X_B \to E^\star_B$, $u_B : \tilde{X}_B \to X_B$ and $v_B : \tilde{X}_B \to E^\star_B$ induced by p, u and v are translation maps.

Let $\mathbf{dev}_B : \tilde{X}_B \to \mathbb{C}$ be a developing map of $\tilde{X}_B$ (and thus also for X_B and $E^\star_B$) and $\mathbf{der}_B : \mathrm{Aff}^+(\tilde{X}_B) \to \mathrm{GL}_2(\mathbb{R})$ the corresponding derived map.
The proof of the following Remark 5 shows that the affine group of an origami surface X_B does not depend (up to conjugacy) on the choice of the matrix B.

Remark 5. Let B, B' be in $\mathrm{SL}_2(\mathbb{R})$. Then

$$\mathrm{Aff}^+(X_B(O)) \cong \mathrm{Aff}^+(X_{B'}(O)) \text{ and } \Gamma(X_{B'}(O)) = B'B^{-1}\Gamma(X_B(O))BB'^{-1}.$$

Proof. The map $\varphi : X_B(O) \to X_{B'}(O)$ that is topologically the identity on X is an affine diffeomorphism and induces the group isomorphism:

$$\mathrm{Aff}^+(X_B(O)) \to \mathrm{Aff}^+(X_{B'}(O)), f \mapsto \varphi \circ f \circ \varphi^{-1}.$$

Since $\mathbf{der}(\varphi) = B'B^{-1}$, we have $\mathbf{der}(\varphi f \varphi^{-1}) = B'B^{-1}\mathbf{der}(f)BB'^{-1}$ □

Since the Veech group depends only up to conjugacy on the choice of B, we will restrict to the case of $B = I$, the identity matrix. If not stated otherwise, we will denote $\tilde{X} := \tilde{X}_I$, $\mathbf{der} := \mathbf{der}_I$, $\mathbf{dev} := \mathbf{dev}_I$, $X := X_I$, $E := E_I$, $\Lambda := \Lambda_I$, $E^\star := E^\star_I$, $\mu := \mu_I$ and $\Gamma(O) := \Gamma(X_I(O))$.
By the uniformization theorem there exists a biholomorphic map $\delta : \mathbb{H} \to \tilde{X} = \tilde{X}_I$, where $\mathbb{H}$ is the complex upper half plane. $\mathbb{H}$ becomes via δ a translation surface. We will identify in the following $\mathbb{H}$ with $\tilde{X} = \tilde{X}_I$.

Proposition 6. *Let $O = (p : X \to E^\star)$ be an origami and $\mathbb{H}$ be the upper half plane, endowed with the translation structure induced by O as above. Then we have:*

(1) *$\Gamma(O)$ is a subgroup of $\Gamma(\mathbb{H})$.*
(2) *$\Gamma(E^\star) = \Gamma(\mathbb{H}) = \mathrm{SL}_2(\mathbb{Z})$.*
(3) *Let f be in $\mathrm{Aff}^+(X)$. Then f descends via p to some $\bar{f} \in \mathrm{Aff}^+(E^\star)$ and Diagram 1 is commutative with $A := \mathbf{der}(f)$, with $\hat{f}$ some lift of f to $\mathbb{H}$ and with some $b \in \mathbb{Z}^2$.*

$$\begin{tikzcd}
 & \mathbb{H} \arrow[rrr, "\hat{f}"] \arrow[dl, "h"'] \arrow[d, "\mathbf{dev}"] \arrow[ddr, "u"] \arrow[drrr, phantom, "(A)"] & & & \mathbb{H} \arrow[d, "\mathbf{dev}"] \arrow[dr, "h"] \arrow[ddl, "u"'] & \\
\mathbb{C}-\Lambda \arrow[r, phantom, "\subset"] \arrow[ddrr, "w"] & \mathbb{C} \arrow[rrr, "z\mapsto Az+b"] & & & \mathbb{C} \arrow[r, phantom, "\supset"] & \mathbb{C}-\Lambda \arrow[ddll, "w"'] \\
 & & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] & & \\
 & & E^\star \arrow[r, "\bar{f}"] & E^\star & &
\end{tikzcd}$$

Diagram 1

Proof.
1.: Let f be in $\mathrm{Aff}^+(X)$ and $\hat{f}$ be some lift of f via u. Since the translation structure on $\mathbb{H}$ is lifted via u, $\hat{f}$ is also affine and $\mathbf{der}(\hat{f}) = \mathbf{der}(f)$. Hence, $\Gamma(O) \subseteq \Gamma(\mathbb{H})$.

2.: Let $\mathbb{C} \to E$ be the universal covering and $w : \mathbb{C} - \Lambda \to E^\star$ its restriction to $\mathbb{C} - \Lambda$. Since $v = p \circ u$ is the universal covering of $E^\star$, there is an unramified covering $h : \mathbb{H} \to \mathbb{C} - \Lambda$, such that $w \circ h = v = p \circ u$. But since the structure on $\mathbb{H}$ was obtained by lifting the translation structure on $E^\star$ via v, this map h is locally a chart of $\mathbb{H} = \tilde{X}_I$. Thus, the map h is a developing map and the image of this developing map $\mathbf{dev}$ is $\mathbb{C} - \Lambda$.
Now, let A be in $\Gamma(\mathbb{H})$, hence $A = \mathbf{der}(\hat{f})$ for some $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$. By the definition of $\mathbf{der}$ and $\mathbf{dev}$ Part (A) of Diagram 1 is commutative for some $b \in \mathbb{Z}^2$, i. e.

$$(z \mapsto Az + b) \circ \mathbf{dev} = \mathbf{dev} \circ \hat{f}.$$

Since the image of $\mathbf{dev}$ is in $\mathbb{C} - \Lambda$, the map $z \mapsto Az + b$ respects $\Lambda = \mathbb{Z}^2$. Thus, A is in $\mathrm{SL}_2(\mathbb{Z})$. Hence, we have: $\Gamma(\mathbb{H}) \subset \mathrm{SL}_2(\mathbb{Z})$.
Conversely, taking a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ the map $z \mapsto Az$ descends to an affine diffeomorphism $\bar{f} \in \mathrm{Aff}^+(E^\star)$. This can be lifted to some $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ with $\mathbf{der}(\hat{f}) = A$. Thus, we have: $\mathrm{SL}_2(\mathbb{Z}) \subset \Gamma(\mathbb{H})$.
Using the same arguments it follows that also $\Gamma(E^\star) = \mathrm{SL}_2(\mathbb{Z})$.

3.: Let $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ be some lift of f to $\mathbb{H}$. By the proof of (2) it follows that $\hat{f}$ descends via $w \circ h = v$ to some $\bar{f} \in \mathrm{Aff}^+(E^\star)$ and that Diagram 1 is commutative. □

From (1) and (2) of Proposition 6 we see in particular that the Veech group $\Gamma(O)$ of an origami O is always a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. It follows from [Gutkin/Judge 2000, Thm. 5.5], that it has finite index in $\mathrm{SL}_2(\mathbb{Z})$. This result will play a crucial role in section 3.3.

An immediate consequence of Proposition 6 is

Corollary 7.

$\Gamma(O) = \{A \in \mathrm{SL}_2(\mathbb{Z}) | A = \mathbf{der}(\hat{f})$ for some $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ that descends to X via $u\}$.

To prove Proposition 1 from Corollary 7 we have to state a condition for $\hat{f}$ in $\mathrm{Aff}^+(\mathbb{H})$ to descend via u to some $f \in \mathrm{Aff}^+(X)$.

2.2. When does an element in $\mathbf{Aff}^+(\mathbb{H})$ descend to X?

Recall that $H = \mathrm{Gal}(\mathbb{H}/X) \subset F_2 = \mathrm{Gal}(\mathbb{H}/E^\star) \subseteq \mathrm{PSL}_2(\mathbb{R})$ (Notation 3). We define the group homomorphism

$$\begin{array}{rrcl} \star : & \mathrm{Aff}^+(\mathbb{H}) & \to & \mathrm{Aut}^+(F_2) \\ & \hat{f} & \mapsto & (\hat{f}_\star : \sigma \mapsto \hat{f} \circ \sigma \circ \hat{f}^{-1}) \end{array}$$

Remark that

$$F_2 = \mathrm{Gal}(\mathbb{H}/E^\star) = \{\hat{f} \in \mathrm{Aff}^+(\mathbb{H}) | \mathbf{der}(\hat{f}) = I\}. \qquad [1]$$

The map $\star$ is well defined, since $\hat{f} \circ \sigma \circ \hat{f}^{-1}$ is again affine with the derivative $\mathbf{der}(\hat{f}) \cdot I \cdot \mathbf{der}(\hat{f})^{-1} = I$ and thus in F_2.

Lemma 8. *We have the following properties of $\star$:*

(1) *The following two sequences are exact and the diagram is commutative:*

$$\begin{tikzcd}
1 \arrow[r] & F_2 \arrow[r] \arrow[d, "\alpha", "\cong"'] \arrow[dr, phantom, "(A)"] & \mathrm{Aff}^+(\mathbb{H}) \arrow[r, "\mathbf{der}"] \arrow[d, "\star", "\cong"'] \arrow[dr, phantom, "(B)"] & \mathrm{SL}_2(\mathbb{Z}) \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{Inn}(F_2) \arrow[r] & \mathrm{Aut}^+(F_2) \arrow[r] & \mathrm{Out}^+(F_2) \arrow[r] \arrow[u, "\beta"', "\cong"] & 1
\end{tikzcd}$$

Diagram 2

Here, $\mathrm{Inn}(F_2)$ *is the group of inner automorphisms of* F_2*,* α *is the natural isomorphism* $F_2 \to \mathrm{Inn}(F_2), x \mapsto (y \mapsto xyx^{-1})$, $\beta : \mathrm{Out}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ *is the group isomorphism induced by the natural homomorphism:*

$$\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z}),\ \varphi \mapsto A := \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a *is the number of* x *appearing in* $\varphi(x)$*,* b *the number of* x *appearing in* $\varphi(y)$*,* c *the number of* y *in* $\varphi(x)$ *and* d *the number of* y *in* $\varphi(y)$ *(see* [Lyndon/Schupp 1977, I 4.5, p.25]*). Recall that for the canonical projection* $\mathrm{proj} : F_2 \to \mathbb{Z}^2$ *sending* x *to* $(1,0)^t$ *and* y *to* $(0,1)^t$ *one has:*

$$\forall \varphi \in \mathrm{Aut}^+(F_2), A := \hat{\beta}(\varphi) \qquad \mathrm{proj} \circ \varphi = (z \mapsto A \cdot z) \circ \mathrm{proj}. \qquad [2]$$

(2) *An element* $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ *descends to* X *via* p *iff* $\hat{f}_\star(H) = H$*.*

Proof.
1.:

The exactness of the first sequence follows by Equation 1 and by Proposition 6. The exactness of the second sequence is true by the definition of $\mathrm{Out}^+(F_2)$.
The commutativity of Part (A) of the Diagram is true by definition of $\star$. We prove now the commutativity of Part (B):
We have chosen the isomorphism $F_2 = F_2(x,y) \cong \mathrm{Gal}(\mathbb{H}/E^\star)$ and the translation structure on $E^\star = E_I^\star$ in such a way that:

$$\mathbf{aff}(x) = (z \mapsto z + \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \text{ and } \mathbf{aff}(y) = (z \mapsto z + \begin{pmatrix} 0 \\ 1 \end{pmatrix}).$$

Thus, $\mathbf{aff}|_{F_2}(= \mathbf{hol})$ is the natural projection $\mathrm{proj} : F_2 \to \mathbb{Z}^2$. Here we identify the group of translations of $\mathbb{C}$ along some vector in $\mathbb{Z}^2$ canonically with $\mathbb{Z}^2$.
Consider the following diagram:

$$\begin{tikzcd}
F_2 \arrow[r, "\hat{f}_\star"] \arrow[d, "\mathrm{proj}"'] & F_2 \arrow[d, "\mathrm{proj}"] \\
\mathbb{Z}^2 \arrow[r, "z \mapsto A \cdot z"] & \mathbb{Z}^2
\end{tikzcd}$$

Diagram 3

Diagram 3 is commutative with $A := \mathbf{der}(\hat{f})$:
Let σ be in $F_2 = \mathrm{Gal}(\mathbb{H}/E^\star)$. We have to show that $\mathrm{proj}(\hat{f}_\star(\sigma)) = A \cdot \mathrm{proj}(\sigma)$.

We have $\mathbf{aff}(\sigma) = (z \mapsto z + c)$ and $\mathbf{aff}(\hat{f}) = (z \mapsto Az + b)$ for some $b, c \in \mathbb{Z}^2$. Thus we get:

$$\mathrm{proj}(\hat{f}_\star(\sigma)) = \mathbf{aff}(\hat{f}_\star(\sigma)) = \mathbf{aff}(\hat{f})\mathbf{aff}(\sigma)\mathbf{aff}(\hat{f}^{-1}) = (z \mapsto z + Ac).$$

Hence, Diagram 3 is commutative with $A = \mathbf{der}(\hat{f})$.
To conclude we use that Diagram 3 is also commutative with $A = \hat{\beta}(\hat{f}_\star)$ (see equation [2]). Thus, $\mathbf{der}(\hat{f}) = \hat{\beta}(\hat{f}_\star)$ and (B) is commutative.
Finally, α and β are both isomorphisms, thus $\star$ is also an isomorphism.

2.: $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ descends to X via $p \Leftrightarrow$ for all $z \in \mathbb{H}, \sigma \in H = \mathrm{Gal}(\mathbb{H}/X)$ there is some $\tilde{\sigma}_{z,\sigma} \in H$ such that $\tilde{\sigma}_{z,\sigma}(\hat{f}(z)) = \hat{f}(\sigma(z))$.
For $\tilde{\sigma} := \hat{f}_\star(\sigma)$ we have by definition of $\hat{f}_\star$: $\tilde{\sigma}(\hat{f}(z)) = \hat{f}(\sigma(z))$ for all $z \in \mathbb{H}$. Since F_2 operates fixpointfree on $\mathbb{H}$ it follows from the last equation that $\tilde{\sigma}_{z,\sigma}$ has to be equal to $\tilde{\sigma} = \hat{f}_\star(\sigma)$. On the other hand, $\tilde{\sigma}_{z,\sigma}$ has to be in H. This proves (2). □

Now Proposition 1 follows from Corollary 7 and Lemma 8. □

As result of Proposition 1 we get: In order to check whether $A \in \mathrm{SL}_2(\mathbb{Z})$ is in $\Gamma(O)$, we have to check if there exists a lift $\gamma_A \in \mathrm{Aut}^+(F_2)$ of A (i.e. a preimage of A under $\hat{\beta}$) that fixes H. The following Corollary translates this into a finite problem that can be left to a computer.

Corollary 9. *(to Proposition 1)*
Let $O = (p : X \to E^\star)$ be an origami of degree d, $F_2 = \mathrm{Gal}(\mathbb{H}/E^\star)$, $H = \mathrm{Gal}(\mathbb{H}/X)$ as above. Let $h_1, \ldots, h_k$ be generators of H and $\sigma_1, \ldots, \sigma_d$ a system of right coset representatives of $H\backslash F_2$ (denote the right coset $H \cdot \sigma_i$ by $\bar{\sigma}_i$).
Further let $\gamma_A^0 \in \mathrm{Aut}^+(F_2)$ be some fixed lift of $A \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$A \in \Gamma(O) \Leftrightarrow \exists\, i \in \{1, \ldots, d\} \textit{ such that } \bar{\sigma}_i \cdot \gamma_A^0(h_j) = \bar{\sigma}_i \textit{ for all } j \in \{1, \ldots, k\}.$$

Proof. Let γ_A be another lift of A. Thus $\gamma_A^0 = \sigma^{-1} \cdot \gamma_A \cdot \sigma$ for some $\sigma \in F_2$ and we have for all h in H:

$$\gamma_A(h) \in H \Leftrightarrow \sigma \cdot \gamma_A^0(h) \cdot \sigma^{-1} \in H \Leftrightarrow H \cdot \sigma \cdot \gamma_A^0(h) = H \cdot \sigma \Leftrightarrow \bar{\sigma} \cdot \gamma_A^0(h) = \bar{\sigma}$$

Hence, the claim follows from Proposition 1. □

3. The algorithm

Let $O = (p : X \to E^\star)$ be a given origami of degree d. In this section we present our algorithm that determines the Veech group $\Gamma(O)$. We have subdivided this description into four parts: In 3.1 we describe how to find some lift $\gamma_A \in \mathrm{Aut}^+(F_2)$ for any matrix A in $\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Out}^+(F_2)$, in 3.2 we show how to decide whether a given matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ is in $\Gamma(O)$, in 3.3 we give an algorithm that determines generators and a system of coset representatives of $\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$, and finally in 3.4 we state how to calculate the genus and the points at infinity of the corresponding *Veech curve* $\mathbb{H}/\Gamma(O)$.
In order to illustrate the algorithm we will use the example $O = L(2, 3)$.

Example 10. (The Origami $O = L(2,3)$)

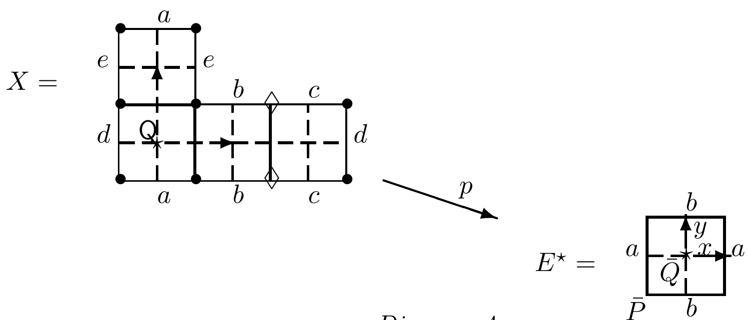


Diagram 4

In Example 10 the edges labelled with the same letters are glued together. This way X becomes a surface of genus 2. The squares describe the covering map to $E^\star$. The point $\bar{P} \in E$ (at infinity) has 2 preimages on the surface X (the points $\bullet$ and $\Diamond$), the degree d of p is 4.
We identify $F_2 = \mathrm{Gal}(\mathbb{H}/E^\star)$ with the fundamental group of $E^\star$ (with base point $\bar{Q}$) and $H = \mathrm{Gal}(\mathbb{H}/X)$ with the fundamental group of X (with base point Q). The projection of the closed paths on X to $E^\star$ defines the embedding of H into F_2, x and y are the fixed generators of F_2 on $E^\star$. Since the $L(2,3)$-shape is simply connected, the generators of H are obtained by the identifications of the edges. Thus, $H = < x^3, x^2yx^{-2}, xyx^{-1}, yxy^{-1}, y^2 >$. The index $[F_2 : H]$ is equal to $d = 4$.

3.1. **Lifts from $\mathrm{SL}_2(\mathbb{Z})$ to the automorphism group of F_2.**

Let
$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
We will use the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T and that $S^{-1} = -S$ and $T^{-1} = -STSTS$. Thus, every $A \in \mathrm{SL}_2(\mathbb{Z})$ can be written as $A = W(S,T)$ or $A = -W(S,T)$, where W is a word in the letters S and T.
The homomorphisms
$$\begin{aligned} \gamma_S &: F_2 \to F_2 \text{ defined by } \gamma_S(x) = y \text{ and } \gamma_S(y) = x^{-1}, \\ \gamma_T &: F_2 \to F_2 \text{ defined by } \gamma_T(x) = x \text{ and } \gamma_T(y) = xy \quad \text{ and} \\ \gamma_{-I} &: F_2 \to F_2 \text{ defined by } \gamma_{-I}(x) = x^{-1} \text{ and } \gamma_{-I}(y) = y^{-1} \end{aligned}$$

are in $\mathrm{Aut}^+(F_2)$ with $\hat{\beta}(\gamma_S) = S$, $\hat{\beta}(\gamma_T) = T$ and $\hat{\beta}(\gamma_{-I}) = -I$, where the morphism $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ is the projection defined in 2.2 (Lemma 8).
Hence, for $A = \pm W(S,T)$ the automorphism $\gamma_A := \pm W(\gamma_S, \gamma_T) \in \mathrm{Aut}^+(F_2)$ is a lift of A. Hereby we denote $-W(\gamma_S, \gamma_T) := \gamma_{-I} \circ W(\gamma_S, \gamma_T)$.

In order to find a word W such that $A = W(S,T)$ or $A = -W(S,T)$ we will define a sequence $A_1 := A$, A_2, ..., A_N such that (for $1 \le n < N$)
$$A_{n+1} = A_n \cdot T^{-k_n} \cdot S \text{ (with } k_n \in \mathbb{Z}) \text{ and } A_N = \pm T^{\pm b_N} \text{ (with } b_N \in \mathbb{Z}).$$
From this we get that $A = \pm T^{\pm b_n} \cdot (-S) \cdot T^{k_{n-1}} \cdot \ldots \cdot (-S) \cdot T^{k_1}$. We will conclude using that $T^{-1} = -STSTS$.

These considerations give rise to the following algorithm, in which we denote

$$A_n =: \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \text{ with } a_n, b_n, c_n, d_n \in \mathbb{Z}.$$

Algorithm for finding a lift in $\mathbf{Aut}^+(F_2)$:

Given: $A \in \mathrm{SL}_2(\mathbb{Z})$.

$n := 1; A_1 := A$.

(1) If $c_n \neq 0$ find $k_n \in \mathbb{Z}$, such that

$$A_{n+1} := A_n T^{-k_n} S \text{ fulfills } |c_{n+1}| < |c_n|.$$

$k_n := d_n \operatorname{div} c_n$ does this job: $d_n = k_n c_n + r_n$ with $r_n \in \{0, 1, \ldots, |c_n| - 1\}$

$$\Rightarrow A_{n+1} = \begin{pmatrix} -a_n k_n + b_n & -a_n \\ r_n & -c_n \end{pmatrix}.$$

Increase n by 1.

(2) Iterate Step (1) until $c_n = 0$. Thus

$$\begin{aligned} A_n &= \begin{pmatrix} \pm 1 & b_n \\ 0 & \pm 1 \end{pmatrix} = \pm T^{\pm b_n} \text{ and} \\ A &= \pm T^{\pm b_n} \cdot (-S) \cdot T^{k_{n-1}} \cdot \ldots \cdot (-S) \cdot T^{k_1} =: \pm \tilde{W}(S, T, T^{-1}). \end{aligned}$$

(3) Replace in $\tilde{W}$ each T^{-1} by $-STSTS$
$\Rightarrow$ Word W in S and T with $A = W(S,T)$ or $A = -W(S,T)$.

(4) Compute $\gamma_A := W(\gamma_S, \gamma_T)$ or $\gamma_A := -W(\gamma_S, \gamma_T)$.

Result: $\gamma_A \in \mathrm{Aut}(F_2)$ with $\hat{\beta}(\gamma_A) = A$.

Example 11.

$$\begin{aligned} \begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix} = -T^2 S T^3 S T S \quad &\Rightarrow \quad \gamma_A^0 = \gamma_{-I} \gamma_T^2 \gamma_S \gamma_T^3 \gamma_S \gamma_T \gamma_S \\ &\Rightarrow \quad \gamma_A^0 : \ x \mapsto x^{-2} y^{-1} x^{-2} y^{-1} x^{-2} y^{-1} x y x^2, \quad y \mapsto x^{-1} y x^2 y x^2 y x^2 \end{aligned}$$

3.2. Decide whether A is in the Veech Group $\Gamma(O)$.

Let A be in $\mathrm{SL}_2(\mathbb{Z})$. We want to decide whether A is in $\Gamma(O)$ or not. As in Corollary 9 let $h_1, \ldots, h_k$ be generators of $H = \mathrm{Gal}(\mathbb{H}/X) \subseteq F_2 = \mathrm{Gal}(\mathbb{H}/E^\star)$, $\sigma_1, \ldots, \sigma_d$ a system of right coset representatives of H in F_2 ($\bar{\sigma}_i := H \cdot \sigma_i$) and γ_A^0 some fixed lift of A in $\mathrm{Aut}^+(F_2)$.
Corollary 9 suggests how to build the algorithm:

$$A \in \Gamma(O) \Leftrightarrow \exists i \in \{1, \ldots, d\} \text{ such that } \forall j \in \{1, \ldots, k\} \quad \bar{\sigma}_i \cdot \gamma_A^0(h_j) = \bar{\sigma}_i.$$

Hence, the *main step* will be to decide for some $\tau \in F_2$ whether

$$\bar{\sigma}_i \cdot \tau = \bar{\sigma}_i.$$

In order to do this we present the origami O as directed graph G with edges labelled by x and y (see Figure 5). The cosets $\bar{\sigma}_1, \ldots, \bar{\sigma}_d$ are the vertices of G. Each vertex $\bar{\sigma}_i$ is start point of one x-edge and one y-edge. The endpoint is $\overline{\sigma_i \cdot x}$ and $\overline{\sigma_i \cdot y}$,

respectively.

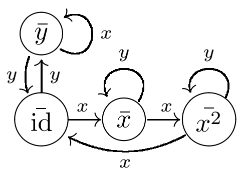


Figure 5: Graph for $O = L(2,3)$.

Writing $\tau \in F_2$ as word in x,y,x^{-1} and y^{-1} defines a not necessarily oriented path in G starting at the vertex $\bar{\sigma_i}$ with end point $\bar{\sigma_i} \cdot \tau$. We have:

$$\bar{\sigma_i} \cdot \tau = \bar{\sigma_i} \Leftrightarrow \text{ this path is closed.}$$

Thus we get the following algorithm.

Algorithm for deciding whether A is in $\Gamma(O)$:

Given: $A \in \mathrm{SL}_2(\mathbb{Z})$.

Calculate some lift $\gamma_A^0 \in \mathrm{Aut}^+(F_2)$ of A (see 3.1).
For $j = 1$ to k do: $\tilde{h}_j := \gamma_A^0(h_j)$.
result := false.
for $i = 1$ to d do

help := true.
for $j = 1$ to k do: if $\bar{\sigma_i} \cdot \tilde{h}_j \neq \bar{\sigma_i}$ (main step, see above) then help := false.
if help = true then result := true.

Result: If the variable 'result' is true, then $A \in \Gamma(O)$, else $A \notin \Gamma(O)$.

Example 12. (for $O = L(2,3)$)
Let $A := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Take the lift:

$$\gamma_A^0 : \ x \mapsto xyxyx^{-1} =: u \quad y \mapsto xyxyx^{-1}y^{-1}x^{-1} =: v$$

Generators of H (see Ex. 10) are:

$$h_1 := x^3, h_2 := xyx^{-1}, h_3 := x^2yx^{-2}, h_4 := yxy^{-1}, h_5 := y^2.$$

For example $\bar{\mathrm{id}} \cdot \gamma_A^0(h_2) = \bar{\mathrm{id}} \cdot uvu^{-1} = \bar{x}vu^{-1} = \bar{x^2}u^{-1} = \bar{x^2} \ \Rightarrow \ \gamma_A^0(H) \neq H$.
But one has: $\bar{x} \cdot \gamma_A^0(h_i) = \bar{x} \ \forall i \in \{1, \ldots, 5\}$.
$\Rightarrow \gamma_A(H) = H$ for $\gamma_A = x \cdot \gamma_A^0 \cdot x^{-1}$ and $A \in \Gamma(O)$.

3.3. Generators and Coset Representatives of $\Gamma(O)$.

Let $\bar{\Gamma}(O)$ be the *projective Veech group*, i.e. the image of $\Gamma(O)$ under the projection of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{PSL}_2(\mathbb{Z})$. We first give an algorithm that calculates a list **Gen** of generators and a list **Rep** of right coset representatives of $\bar{\Gamma}(O)$ in $\mathrm{PSL}_2(\mathbb{Z})$, then we determine $\Gamma(O)$. The way how we proceed is based on the Reidemeister-Schreier method ([Lyndon/Schupp 1977], II.4).
We denote by $\bar{A}$ the image of an element $A \in \mathrm{SL}_2(\mathbb{Z})$ under the projection to $\mathrm{PSL}_2(\mathbb{Z})$ and, conversely, denote for $\bar{A}$ in $\mathrm{PSL}_2(\mathbb{Z})$ by A some lift of $\bar{A}$. Moreover, we write $A \sim B$ (respectively $\bar{A} \sim \bar{B}$) if they are in the same coset, i.e. $\Gamma(O) \cdot A = \Gamma(O) \cdot B$ (respectively $\bar{\Gamma}(O) \cdot \bar{A} = \bar{\Gamma}(O) \cdot \bar{B}$).

Each element of $\mathrm{PSL}_2(\mathbb{Z})$ can be presented as word in $\bar{S}$ and $\bar{T}$. We use the directed infinite tree shown in Figure 6: The vertices $v_0, v_1, v_2, \ldots$ of the tree are labelled by elements of $\mathrm{PSL}_2(\mathbb{Z})$. The root v_0 is labelled by $\bar{I}$, the image of the identity matrix. Each vertex is starting point of two edges, one labelled by $\bar{S}$, one labelled by $\bar{T}$. Each element of $\mathrm{PSL}_2(\mathbb{Z})$ occurs as label of at least one vertex. Starting with v_0 we will visit each vertex v (with label $\bar{B}$) and check if it is not yet represented by the list **Rep**. In this case we will add it to **Rep**. Otherwise for each $\bar{D}$ in **Rep** that is in the same coset as $\bar{B}$, we add $\bar{B} \cdot \bar{D}^{-1}$ to the list **Gen** of generators.

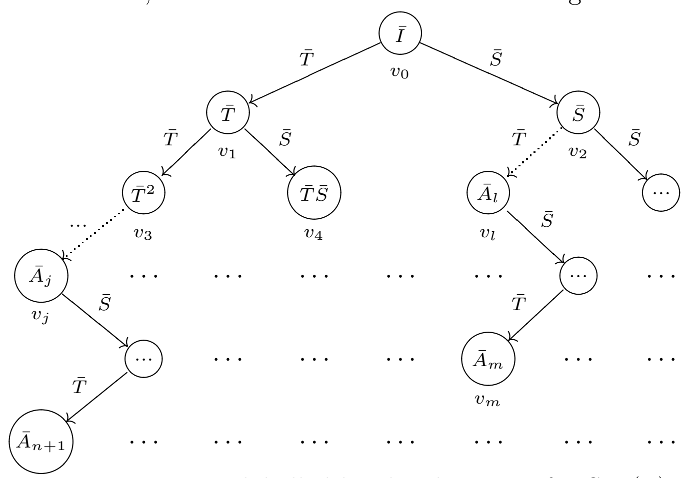


Figure 6: Tree labelled by the elements of $PSL_2(\mathbb{Z})$

We will first give the algorithm and then proof that the lists **Gen** and **Rep** that are calculated are what they should be.

Algorithm for Calculating $\bar{\Gamma}(O)$:

Given: Origami O.

Let **Rep** and **Gen** be empty lists.
Add $\bar{I}$ to **Rep**. $\bar{A} := \bar{I}$.
Loop:
$B := A \cdot T$, $C := A \cdot S$
//Check whether $\bar{B}$ is already represented by **Rep** and add, if there occur some,
//the found generators to **Gen**:
    For each $\bar{D}$ in **Rep**, check whether $B \cdot D^{-1}$ is in $\Gamma(O)$ or $-B \cdot D^{-1}$ is in $\Gamma(O)$. If so, add $\bar{B} \cdot \bar{D}^{-1}$ to **Gen**.
    If none is found, add $\bar{B}$ to **Rep**.

Do the same for C instead of B.
If there exists a successor of $\bar{A}$ in **Rep**, let $\bar{A}$ be now this successor and go to the beginning of the loop.
If not, finish the loop.

Result: **Gen**: list of generators of $\bar{\Gamma}(O)$, **Rep**: list of coset representatives in $\mathrm{PSL}_2(\mathbb{Z})$.

Remark 13.

(1) Any two elements of **Rep** belong to different cosets.
(2) The algorithm stops after finitely many steps.
(3) In the end each coset is represented by a member of **Rep**.
(4) In the end $\bar{\Gamma}(O)$ is generated by the elements of **Gen**.

Proof.
1.: The statement follows by induction. It is true in the beginning, since **Rep** contains only $\bar{I}$. After passing through the loop it is still true, since $\bar{B}$ (respectively $\bar{C}$) is only added if $\bar{B} \cdot \bar{D}^{-1}$ (resp. $\bar{C} \cdot \bar{D}^{-1}$) is not in $\bar{\Gamma}(O)$ for all $\bar{D}$ in **Rep**.

2.: Follows from 1, since $\bar{\Gamma}(O)$ has finite index in $\mathrm{PSL}_2(\mathbb{Z})$ ([Gutkin/Judge 2000], Thm. 5.5).

3.: Let $\bar{A}$ be an arbitrary element of $\mathrm{PSL}_2(\mathbb{Z})$. There is at least one vertex in the tree that is labelled by $\bar{A}$. Denote the vertices by v_0, v_1, v_2, $\ldots$ as in Figure 6 and their labels by $\bar{A}_0$, $\bar{A}_1$, $\bar{A}_2$, $\ldots$, respectively.
We do induction by the numeration n of the vertices:
$\bar{A}_0 = \bar{I}$ is in **Rep**. Suppose for a certain $n \in \mathbb{N}$ all $\bar{A}_k$ with $k \leq n$ are represented by **Rep**.
If A_{n+1} is not itself in **Rep** then consider the path ω from v_0 to v_{n+1} and let v_j be the first vertex on ω that is not in **Rep**. Hence, its predecessor is in **Rep** and $\bar{A}_j$ was checked but not added. Thus, there is some $\bar{A}_l$ $(l < j)$ in **Rep** such that $\bar{A}_j \cdot \bar{A}_l^{-1}$ is in $\bar{\Gamma}(O)$, i.e. $\bar{A}_j \sim \bar{A}_l$.
Let $\hat{\omega}$ be the path from v_j to v_{n+1} and $\bar{D}$ the product of the labels of the edges on $\hat{\omega}$. Then $\bar{A}_{n+1} = \bar{A}_j \cdot \bar{D}$.
Walking 'the same path' as $\hat{\omega}$ starting at v_l (i.e. a path described by the same sequence of $\bar{S}$ and $\bar{T}$) leads to some vertex v_m with $m < n+1$ and label $\bar{A}_m = \bar{A}_l \cdot \bar{D}$.
We have $\bar{A}_{n+1} = \bar{A}_j \cdot \bar{D} \sim \bar{A}_l \cdot \bar{D} = \bar{A}_m$ and by the assumption $\bar{A}_m$ is represented by **Rep**, hence also $\bar{A}_{n+1}$ is.

4.: Let G be the group generated by the elements of **Gen**. We have by construction of the list **Gen** that $G \subseteq \bar{\Gamma}(O)$.
We show again by induction that each label $\bar{A}_n$ in the tree that is in $\bar{\Gamma}(O)$ is also in G. This is true for $n = 0$. Suppose it is true for all $k \leq n$ with a certain $n \in \mathbb{N}$.
If $\bar{A}_{n+1}$ is in $\bar{\Gamma}(O)$, we proceed as in (3) and find some $\bar{A}_j$, $\bar{A}_l$, $\bar{A}_m$ and $\bar{D}$ $(j, l, m < n+1)$ such that $\bar{A}_j$ and $\bar{A}_l$ are in the same coset, $\bar{A}_j \cdot \bar{A}_l^{-1}$ is in the list **Gen** (hence, $\bar{A}_j \cdot \bar{A}_l^{-1} \in G$), $\bar{A}_{n+1} = \bar{A}_j \cdot \bar{D}$ and $\bar{A}_m = \bar{A}_l \cdot \bar{D}$. $\bar{A}_m$ is in the same coset as $\bar{A}_{n+1}$, thus it is an element of $\bar{\Gamma}(O)$. By the assumption $\bar{A}_m$ is then also in G. Hence, we have:

$$\bar{A}_{n+1} = \bar{A}_j \cdot \bar{A}_l^{-1} \cdot \bar{A}_l \cdot \bar{D} = (\bar{A}_j \cdot \bar{A}_l^{-1}) \cdot \bar{A}_m \in G.$$

□

Now - knowing $\bar{\Gamma}(O)$ -, it is easy to determine $\Gamma(O)$. We just have to distinguish the two cases, whether $-I$ is in $\Gamma(O)$ or not.

Algorithm for Calculation of $\Gamma(O)$:

Given: Origami O.

Calculate **Gen** and **Rep**.
Let **Gen**$'$ and **Rep**$'$ be empty lists.
Check, whether $-I \in \Gamma(O)$.
If yes: For each $\bar{A} \in$ **Gen** add A to **Gen**$'$. Add $-I$ to **Gen**$'$.
For each $\bar{A} \in$ **Rep** add A to **Rep**$'$.
If no: For each $\bar{A} \in$ **Gen**, check whether $A \in \Gamma(O)$.
If it is, add A to **Gen**$'$; if it is not, add $-A$ to **Gen**$'$.
For each $\bar{A} \in$ **Rep** add A and $-A$ to **Rep**$'$.

Result: **Gen**$'$: list of generators of $\Gamma(O)$,
Rep$'$: list of right coset representatives of $\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$.

Example 14. (for $O = L(2,3)$)
1) Result of calculating $\bar{\Gamma}(O)$:

Gen:

$$\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix} = \bar{T}^3, \begin{pmatrix}-1 & 3\\ -2 & 5\end{pmatrix} = \bar{T}\bar{S}\bar{T}^2\bar{S}\bar{T}^{-1}\bar{T}^{-1}, \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix} = \bar{T}\bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}, \begin{pmatrix}3 & -5\\ 2 & -3\end{pmatrix} = \bar{T}^2\bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}^{-1}\bar{T}^{-2}$$

is a list of generators of $\bar{\Gamma}(O)$.
Rep :

$$\bar{I},\ \bar{T},\ \bar{S},\ \bar{T}^2,\ \bar{T}\bar{S},\ \bar{S}\bar{T},\ \bar{T}^2\bar{S},\ \bar{T}\bar{S}\bar{T},\ \bar{T}^2\bar{S}\bar{T}$$

is a system of coset representatives of $\bar{\Gamma}(O)$ in $\mathrm{SL}_2(\mathbb{Z})$.
(The algorithm produces more generators (compare example 16). We eliminated redundant ones.)

2) Result of calculating $\Gamma(O)$: $(-I \in \Gamma(O))$

$$\begin{aligned}\mathbf{Gen}' &= \mathbf{Gen} \cup \{-I\}.\\ \mathbf{Rep}': &= I,\ T,\ S,\ T^2,\ TS,\ ST,\ T^2S,\ TST,\ T^2ST\end{aligned}$$

Hence, $\Gamma(O)$ is a subgroup of index 9 in $\mathrm{SL}_2(\mathbb{Z})$.

3.4. Geometrical type of $\mathbb{H}/\bar{\Gamma}(O)$.

The group $\bar{\Gamma}(O)$ is a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ and of finite index ([Gutkin/Judge 2000, Thm. 5.5]), thus it operates as Fuchsian group (via Möbius transformations) on $\mathbb{H}$ and $V := \mathbb{H}/\bar{\Gamma}(O)$ is an affine algebraic curve. It is defined over $\bar{\mathbb{Q}}$ by the Theorem of Belyi: We have a covering from $\mathbb{H}/\bar{\Gamma}(O)$ to $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{A}^1(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}) - \{\infty\}$ ramified at most over the images of i and $\rho = \frac{1}{2} + (\frac{1}{2}\sqrt{3})i$. Thus, by Belyi's theorem the projective curve $\overline{\mathbb{H}/\bar{\Gamma}(O)}$ and hence also C is defined over $\overline{\mathbb{Q}}$.
We want to determine the genus and the number of points at infinity of the curve $\mathbb{H}/\bar{\Gamma}(O)$.

Let $\Delta := \Delta(P_0, P_1, P_\infty)$ be the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$, i.e. the hyperbolic pseudo-triangle with vertices $P_0 := -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $P_1 := \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $P_\infty := i\infty$.
We denote by $\bar{A}$ also the Möbius transformation defined by the matrix A. Then $\bar{T}$ and $\bar{S}$ (as Möbius transformations) send P_0P_∞ to P_1P_∞, respectively P_0P_1 to itself

(fixing i).
Let **Rep** $= \{\bar{A}_1, \ldots, \bar{A}_k\}$ be the system of right coset representatives we calculated in section 3.3. Then

$$F := \bigcup_{i=1}^{k} \bar{A}_i(\Delta)$$

is a simply connected fundamental domain of $\bar{\Gamma}(O)$. The list **Gen** of generators defines how to glue the edges of F to obtain $\mathbb{H}/\bar{\Gamma}(O)$. This way, we get a triangulation of $\mathbb{H}/\bar{\Gamma}(O)$ (compare Figure 7). We calculate the numbers t, e, v of the triangles, the edges and the vertices of this triangulation as described in the following algorithm. Furthermore, the vertices defined by translates of P_∞ are exactly the cusps of $\mathbb{H}/\bar{\Gamma}(O)$. We denote their number by $\hat{v}$. Thus (using the formula of Euler for calculating the genus) we get the following result.

Remark 15. Let t, e, v and $\hat{v}$ be the numbers of triangles, edges, vertices and marked vertices as calculated in the following algorithm. Then $\mathbb{H}/\bar{\Gamma}(O)$ is an affine curve of genus $g = \frac{2-(v-e+t)}{2}$ with $\hat{v}$ cusps.

Algorithm determining the geometrical type of $\mathbb{H}/\bar{\Gamma}(O)$:

Generate a list of triangles $L := \{\bar{A}_1(\Delta), \ldots, \bar{A}_k(\Delta)\}$.
In the triangle $\bar{A}_i(\Delta)$ we call $\bar{A}_i(P_0)\bar{A}_i(P_1)$ (the image of the edge P_0P_1) 'the S-edge'. Similarly, we call $\bar{A}_i(P_1)\bar{A}_i(P_\infty)$ 'the T-edge' and $\bar{A}_i(P_0)\bar{A}_i(P_\infty)$ 'the T^{-1}-edge'.
For each $i, j \in \{1, \ldots, k\}$ identify

- the T-edge of $\bar{A}_j(\Delta)$ with the T^{-1}-edge of $\bar{A}_i(\Delta)$, if $\bar{A}_i \sim \bar{A}_j \cdot \bar{T}$, i.e. if $(\bar{A}_j\bar{T})\bar{A}_i^{-1} \in \bar{\Gamma}(O)$,
- the T^{-1}-edge of $\bar{A}_j(\Delta)$ with the T-edge of $\bar{A}_i(\Delta)$, if $\bar{A}_i \sim \bar{A}_j \cdot \bar{T}^{-1}$ and
- the S-edge of $\bar{A}_j(\Delta)$ with the S-edge of $\bar{A}_i(\Delta)$, if $\bar{A}_i \sim \bar{A}_j \cdot \bar{S}$.
  If an S-edge of some triangle $\bar{A}_j(\Delta)$ is identified with itself (i.e. $i = j$) create an additional triangle: Add a vertex in the middle of this S-edge and add an edge from this new vertex to the opposite vertex in the triangle $\bar{A}_j(\Delta)$. (Compare triangle $\bar{T}^2\bar{S}\bar{T}$ in Figure 7). This is done to get in the end a triangulation of the surface.

$t :=$ number of triangles. $e :=$ number of edges.
$v :=$ number of vertices, $\hat{v} :=$ number of vertices that are endpoints of T-edges.
$g := \frac{2-(v-e+t)}{2}$.

Result: g : genus of $\mathbb{H}/\bar{\Gamma}(O)$ $\hat{v}$: number of vertices at infinity of $\mathbb{H}/\bar{\Gamma}(O)$.

Example 16. (for $O = L(2,3)$)

Rep: $\bar{I}, \bar{T}, \bar{T}^2, \bar{T}^2\bar{S}, \bar{T}^2\bar{S}\bar{T}, \bar{T}\bar{S}, \bar{T}\bar{S}\bar{T}, \bar{S}, \bar{S}\bar{T}$.
Gen: $a := \bar{T}^3, b := \bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}\bar{T}^{-1}, c := \bar{S}\bar{T}^2\bar{S}, d := \bar{T}\bar{S}\bar{T}^2\bar{S}\bar{T}^{-2}$,
$e := \bar{T}\bar{S}\bar{T}^{-2}\bar{S}\bar{T}^{-2}$, $f := \bar{T}^2\bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}\bar{T}^{-2}$

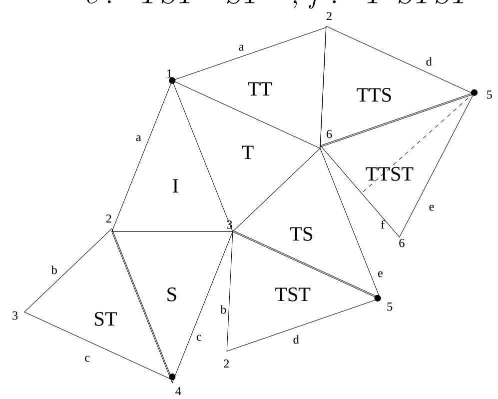


Figure 7: Fundamental domain of $\bar{\Gamma}(L(2,3))$.

Edges with the same letters are glued. In triangle $\bar{T}^2\bar{S}\bar{T}(\Delta)$ an edge and a vertex were added, since the 'S-edge' is glued to itself. Vertices with same numbers are identified. Vertices at infinity are marked by a filled circle.
Thus, $t = 9 + 1, e = 14 + 1, v = 6 + 1, \hat{v} = 3$.

Result: $g = 0, \hat{v} = 3$. Hence,

$$\mathbb{H}/\bar{\Gamma}(L(2,3)) \cong \mathbb{P}^1 - \{0, 1, \infty\}.$$

Proposition 17. $\Gamma(L(2,3))$ *is not a congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Recall from Example 14 that

$$\Gamma(L(2,3)) =< \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix}, \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} > .$$

$\mathbb{H}/\bar{\Gamma}(L(2,3))$ has three cusps represented in Figure 7 by the vertices 1, 4 and 5. T^3, ST^2S^{-1} and $TST^4S^{-1}T^{-1}$ are parabolic elements that correspond to them respectively and the amplitudes are 3, 2 and 4. Hence, the level m of $\Gamma(L(2,3))$ is $\mathrm{lcm}(3,2,4) = 12$ (using notations of [Wohlfahrt 1964]).
Suppose that $\Gamma(L(2,3))$ is a congruence subgroup. By Theorem 2 in [Wohlfahrt 1964] we would have:

$$\Gamma(12) \subseteq \Gamma(L(2,3)). \tag{3.1}$$

Let $p : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ be the natural projection. Then we have

$$p(\bar{\Gamma}(L(2,3))) =< \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{2} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{2} & \bar{0} \end{pmatrix} >= \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z}).$$

Hence Diagram 8 is commutative with $N := \bar{\Gamma}(L(2,3)) \cap \bar{\Gamma}(3)$.

$$\begin{tikzcd}
1 \arrow[r] & \bar{\Gamma}(3) \arrow[r] & \mathrm{PSL}_2(\mathbb{Z}) \arrow[r] & \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z}) \arrow[r] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[u, hook] & \bar{\Gamma}(L(2,3)) \arrow[r] \arrow[u, hook] & \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z}) \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

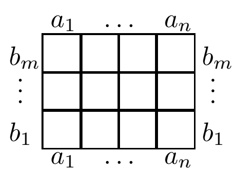
Diagram 8

Since the index $[\mathrm{PSL}_2(\mathbb{Z}) : \bar{\Gamma}(L(2,3))]$ of $\bar{\Gamma}(L(2,3))$ in $\mathrm{PSL}_2(\mathbb{Z})$ is 9 it follows from Diagram 8 that $[\bar{\Gamma}(3) : N] = 9$.
By (3.1) we have: $\bar{\Gamma}(12) \subseteq N \subseteq \bar{\Gamma}(3)$. But $[\bar{\Gamma}(3) : \bar{\Gamma}(12)] = 2^4 \cdot 3$ (using [Shimura 1971], (1.6.2)). Thus $[\bar{\Gamma}(3) : N] = 9$ would have to be a factor of $2^4 \cdot 3$. Contradiction!

□

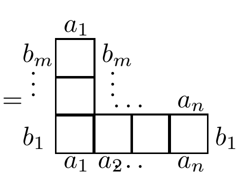
4. Some examples

4.1. **"Trivial Origamis":**

$O =$ 

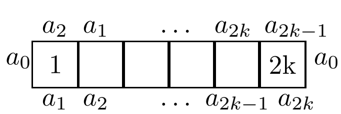
$$\Gamma(O) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | b \equiv 0 \bmod n', c \equiv 0 \bmod m'\}$$
where $t :=\ \gcd(m,n), n' := n/t, m' := m/t$

4.2. **"L-Sequence":**

$L(n,m) =$ 

Origami	Index	Genus	♯ Cusps
$L(2,2)$	3	0	2
$L(2,3)$	9	0	3
$L(2,4)$	18	0	5
$L(2,5)$	36	0	8
$L(2,6)$	54	0	10
$L(2,7)$	108	1	17
$L(3,3)$	9	0	3
$L(4,4)$	54	0	10

4.3. **"Cross - Sequence":**

$O_{2k} =$ 

Origami	Index	Genus	♯ Cusps
O_2	3	0	2
O_4	6	0	3
O_6	12	0	4
O_8	24	0	6
O_{10}	36	0	8
O_{12}	48	0	10
O_{14}	72	1	12
O_{16}	96	2	14

4.4. **Remarks:**

As in Example 10 edges labelled with same letters are glued. The tables in 4.2 and 4.3 itemize for an origami O respectively the index of the projective Veech group $\bar{\Gamma}(O)$ in $\mathrm{PSL}_2(\mathbb{Z})$ and the genus and number of cusps of $\mathbb{H}/\bar{\Gamma}(O)$.
For the example in 4.1, $\Gamma(O)$ can be determined using Proposition 1.
The sequence in 4.2 was introduced to me by Pierre Lochak. The Veech group e.g. of $L(2,2)$ is given also in [Möller 2003]. This sequence is also studied in detail in [Hubert/Lelièvre] and e.g. estimates for the growth of the genus and the number of cusps are obtained. The Veech groups in this sequence are in general not congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ (see Proposition 17).
On the contrary one can show - again using Proposition 1 - that the Veech groups $\Gamma(O_{2k})$ in 4.3 are congruence subgroups for all $k \in \mathbb{N}$. Furthermore the genus of the curve $\mathbb{H}/\Gamma(O_{2k})$ is not bounded.[2]

Only a few general statements about Veech groups of origamis are known yet. There seems to be no obvious relation between the index d of the origami $O = (p : X \to E^\star)$ and the index of its Veech group. In particular, it follows from Proposition 1 that each characteristic subgroup of F_2 defines an origami with Veech group $\mathrm{SL}_2(\mathbb{Z})$. (The smallest, nontrivial example (calculated by Frank Herrlich) is defined by a covering $p : X \to E^\star$ of degree 108.) Hence, there is a cofinal system of origamis having the full group $\mathrm{SL}_2(\mathbb{Z})$ as Veech group.

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[2]Details will be published elsewhere